Chapter S:IV

IV. Search Space Representation

- Problem Solving
- Systematic Search
- Search Space Encoding
- State-Space Representation

- Problem-Reduction Representation
- Choosing a Representation
- Relation to Dynamic Programming

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Problem-Reduction Representation

Problem Solving: Constructing Solutions

Example	Solution Step	Rest Problem
8-Queens	Fix position of next queen.	Positioning of remaining queens on the unattacked fields of the board.
8-Puzzle	Fix next move.	Resulting board configuration that must be transformed into the target configuration.
TSP	Fix next town to visit.	Find trip from this town back to the starting point.
Blocks World	Fix next action (block move).	Resulting block configuration on floor must be transformed into target configuration.

Previous Assumption:

Problem solutions can be described by a linear sequence of applications of (relatively simple) operators that transform a problem into a (hopefully) simpler rest problem.

Q. Is there anything else we may want to do?

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Problem-Reduction Representation

Disadvantage of State-Space Search

A state space search can only determine solutions that are described as paths. Solutions are constructed in a linear way.

Desired solutions can be structured:

- Action-Reaction-Problems

  Solutions are AND-OR trees with AND nodes that represent the reactions of the environment to actions of an agent. A solution describes effective behavior for each of the unpredictable reactions of the environment.

- 2-Player Games

  Solutions are AND-OR trees with AND nodes that represent the reactions of the opponent. A solution describes a game strategy that determines the best move in every game situation.

- Means-end Analysis

  A goal-oriented analysis of the problem leads to a decomposition into a sequence of subproblems. Solutions can be constructed as sequences of solutions for the subproblems.

➜ Independent subproblems can be solved in any order.

➜ Subsolutions can be shared.

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Problem-Reduction Representation

Applicable Heuristic Problem Solving Methods [Zanakis, 1989]

A. Construction

Construction algorithms generate a solution by adding individual components one at a time until a feasible solution is obtained.

➜ OR Graph Search (= State Space Search)

B. Improvement

Improvement heuristics begin with a feasible solution and successively improve it by a sequence of exchanges or mergers in a local search.

➜ Local search algorithms like Hill-Climbing

E. Partitioning (Decomposition [Pearl])

Partitioning algorithms break or 'partition' a problem into smaller subproblems, each of which is solved independently.

➜ AND-OR Graph Search

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Problem-Reduction Representation

Problem Solving: Solution Steps

- Problem Transformation (Simplification)
  A problem is transformed into a single (simpler) problem.

- Problem Decomposition
  A problem is decomposed into a finite number of (sub-) problems.
  - Problem Transformation (Simplification)
    A problem is transformed into a (simpler) problem.
  - Means-End Analysis
    A problem is decomposed into a finite number of (sub-) problems.

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Problem-Reduction Representation

Problem Solving Steps as Different Link Types

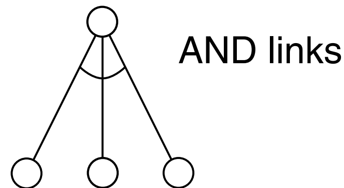
The application of an operator may lead to different kinds of subproblems. A graph that captures such a search space structure has two kinds of links:

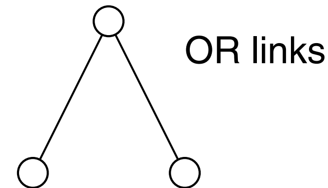
1. AND links.

   Lead to independent subproblems all of which, however, must be solved in order to solve the problem associated with the parent node. AND links of a decomposition step are marked as sibling links.

2. OR links.

   Lead to alternative subproblems one of which has to be solved in order to solve the problem associated with the parent node.





A graph of this type is called problem-reduction graph or AND-OR graph.

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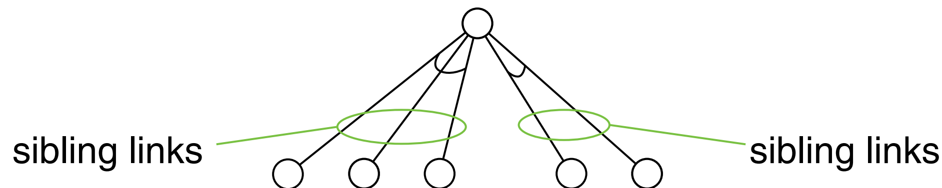
Remarks:

- As in the state-space graph: OR links encode applied operators (problem transformation).
- As in the state-space graph: A sequence of OR links along a path that starts at the root node s defines a solution base.
- Q. Does a path from the root node to a goal node represent a solution to the problem?
- AND links model a problem decomposition. They are employed to decompose a problem according to possible outcomes (as in the counterfeit problem) or to decompose a complex problem into less complex subproblems. [Towers of Hanoi]
- Several possibilities may exist to decompose a given rest problem (at a parent node) into subproblems. The AND links of those subproblems that belong together (sibling links) are hence marked as such.



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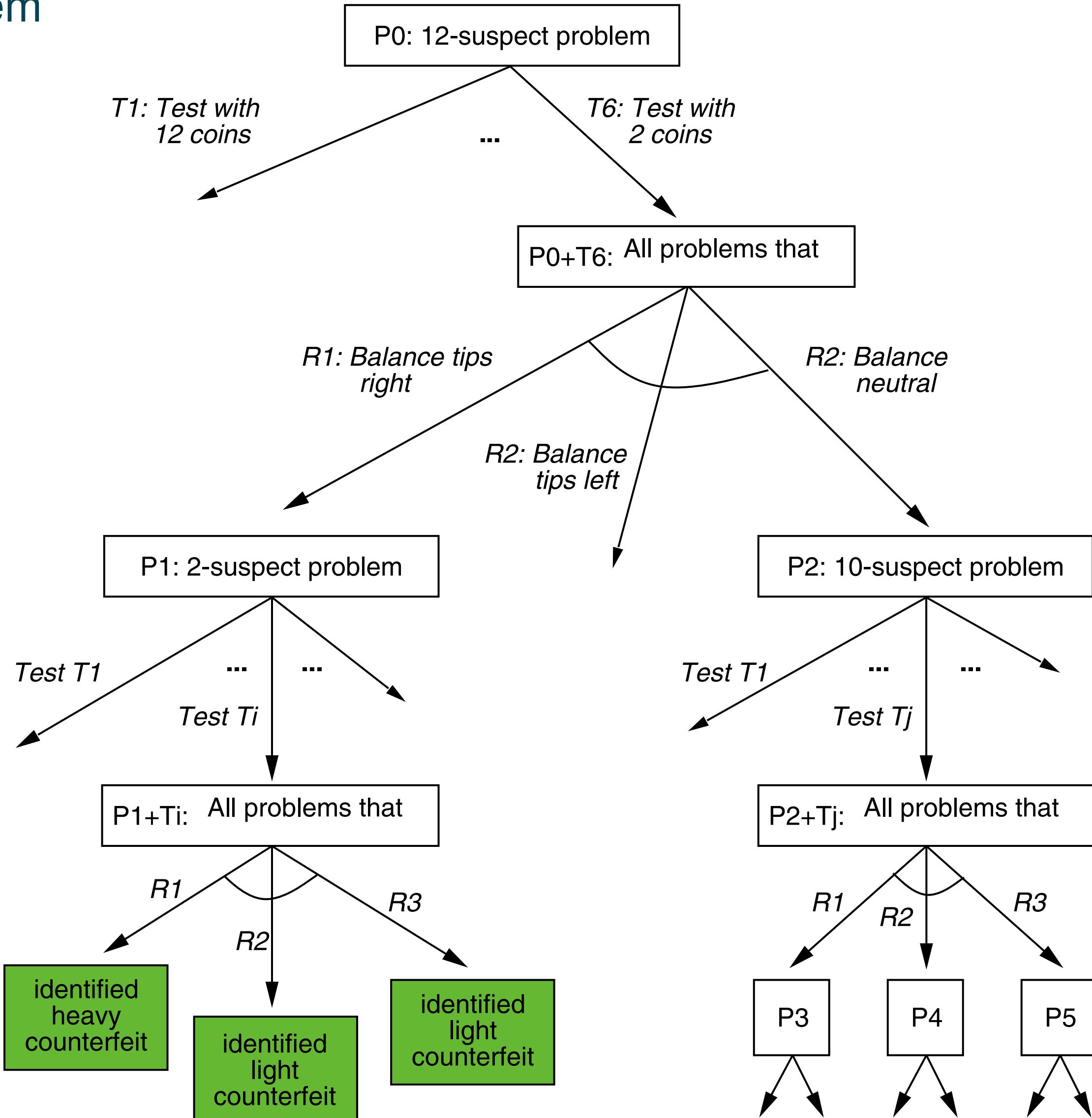
Problem-Reduction Representation

Counterfeit Problem



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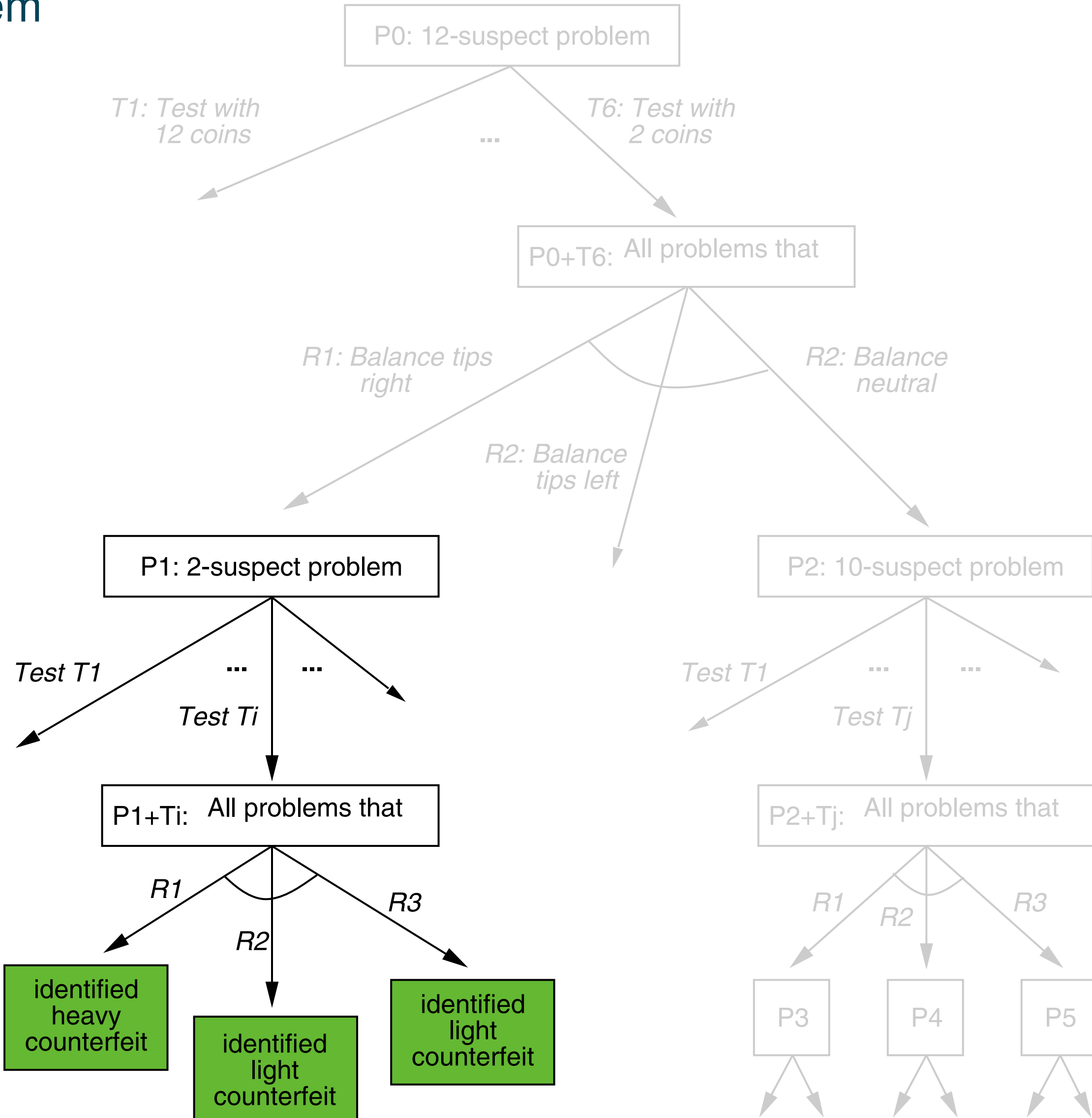
Problem-Reduction Representation

Counterfeit Problem



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Remarks:

- In the 2-suspect problem, we have two coins for which the balance tips one way or another, along with ten remaining coins from which we know that they are honest.
- The 2-suspect problem (for instance) may occur several times in a weighing strategy.
- The 2-suspect problem can be solved independently from all other weighing problems.
- The solution of the 2-suspect problem can be determined, saved, and reused.

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Problem-Reduction Representation

The Different Node types

The application of an operator may lead to different kinds of subproblems. A canonical graph that captures such a search space structure has two kinds of nodes:

1. AND nodes.

   Nodes with outgoing AND links only; more precisely: AND links that belong together, sibling AND links.

   Examples: decisions of an opponent (Keyword: game tree search), random processes in the nature, (artificial) experiments, results of external computations, all kinds of events controlled by a third party

2. OR nodes.

   Nodes with outgoing OR links only.

   Examples: the own decisions, the actual strategy

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Remarks:

- The AND-OR graph of the counterfeit problem has a canonical representation by nature.
- In the AND-OR graph of the counterfeit problem the child nodes of an OR node are AND nodes and vice versa. This alternation of node types is typical for action-reaction problems, e.g. represented in the form of a game tree. [S:VII Game Playing Introduction].

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Problem-Reduction Representation

Search Building Blocks [State-Space]

Definition 5 (Problem-Reduction Graph, AND-OR Graph)

Consider the set of all problems that can be generated by applying either the operators or a problem decomposition to the database.

One obtains the problem-reduction graph or AND-OR graph by

1. connecting each parent node with its generated successors using directed links,
2. labeling those links that belong to an operator with the applied operator,
3. comprising those links that belong to a problem decomposition as sibling links, and
4. introducing additional intermediate nodes and OR-links for each set of sibling AND-links if a node would have more than one set of outgoing sibling links or outgoing AND-links together with outgoing OR-links.

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Remarks:

- Usually, solved-labeling is not applied to an underlying search space graph G. Instead, it is applied to the finite, explored subgraph of G as a termination test: If the root node s is labeled "solved", the explored part of G contains a solution graph.

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Problem-Reduction Representation

Algorithm: solved-labeling

Input: G. A finite, acyclic AND-OR graph.
n. A node in G.
successors(n). Returns the successors of node n.
$\star(n)$. Predicate that is *True* if n is a goal node.

Output: The symbol *True* if a solution exists, *False* otherwise.

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Problem-Reduction Representation

Algorithm: solved-labeling (without labeling)

Input: G. A finite, acyclic AND-OR graph.
n. A node in G.
successors(n). Returns the successors of node n.
$\star(n)$. Predicate that is *True* if n is a goal node.

Output: The symbol *True* if a solution exists, *False* otherwise.

`solved-labeling`$(G, n, \textit{successors}, \star)$

```
1.    IF |successors(n)| = 0 THEN
        IF ⋆(n)
        THEN RETURN(True);
        ELSE RETURN(False);
```


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Problem-Reduction Representation

Algorithm: solved-labeling (without labeling)

Input: G. A finite, acyclic AND-OR graph.
n. A node in G.
successors(n). Returns the successors of node n.
$\star(n)$. Predicate that is *True* if n is a goal node.

Output: The symbol *True* if a solution exists, *False* otherwise.

solved-labeling$(G, n, \textit{successors}, \star)$

```
1.    IF |successors(n)| = 0 THEN
        IF ⋆(n)
        THEN RETURN(True);
        ELSE RETURN(False);
2.    FOREACH n' IN successors(n) DO
        IF solved-labeling(G, n', successors, ⋆)
        THEN
          IF OR_node(n) THEN RETURN(True); // One success is enough.
        ELSE
          IF AND_node(n) THEN RETURN(False); // One fail is one too much.
      ENDDO
```

Problem-Reduction Representation

Algorithm: solved-labeling (without labeling)

Input: G. A finite, acyclic AND-OR graph.
n. A node in G.
successors(n). Returns the successors of node n.
$\star(n)$. Predicate that is *True* if n is a goal node.

Output: The symbol *True* if a solution exists, *False* otherwise.

solved-labeling$(G, n, \textit{successors}, \star)$

```
1.    IF |successors(n)| = 0 THEN
        IF ⋆(n)
        THEN RETURN(True);
        ELSE RETURN(False);
2.    FOREACH n' IN successors(n) DO
        IF solved-labeling(G, n', successors, ⋆)
        THEN
          IF OR_node(n) THEN RETURN(True); // One success is enough.
        ELSE
          IF AND_node(n) THEN RETURN(False); // One fail is one too much.
      ENDDO
3.    IF OR_node(n)
      THEN RETURN(False); // OR_node n has no solvable successor.
      ELSE RETURN(True); // All successors of AND_node n are solvable.
```


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Problem-Reduction Representation

Algorithm: solved-labeling (with labeling)

Input: G. A finite, acyclic AND-OR graph.
n. A node in G.
successors(n). Returns the successors of node n.
$\star(n)$. Predicate that is *True* if n is a goal node.

Output: The symbol *True* if a solution exists, *False* otherwise.

```
solved-labeling(G, n, successors, ⋆)

1.   IF solved(n) THEN RETURN(True);

2.   IF |successors(n)| = 0 THEN
       IF ⋆(n)
       THEN solved(n) = True, RETURN(True);
       ELSE RETURN(False);

3.   FOREACH n' IN successors(n) DO
       IF solved-labeling(G, n', successors, ⋆)
       THEN
         IF OR_node(n) THEN solved(n) = True, RETURN(True);
       ELSE
         IF AND_node(n) THEN RETURN(False);
     ENDDO

4.   IF OR_node(n)
     THEN RETURN(False);
     ELSE solved(n) = True, RETURN(True);
```


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Problem-Reduction Representation

Solution Graphs with Cycles (1)

Cyclic problem-reduction graphs can entail infinite solution graphs, caused by "unfolding":

s

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Problem-Reduction Representation

Solution Graphs with Cycles (1)

Cyclic problem-reduction graphs can entail infinite solution graphs, caused by “unfolding”:

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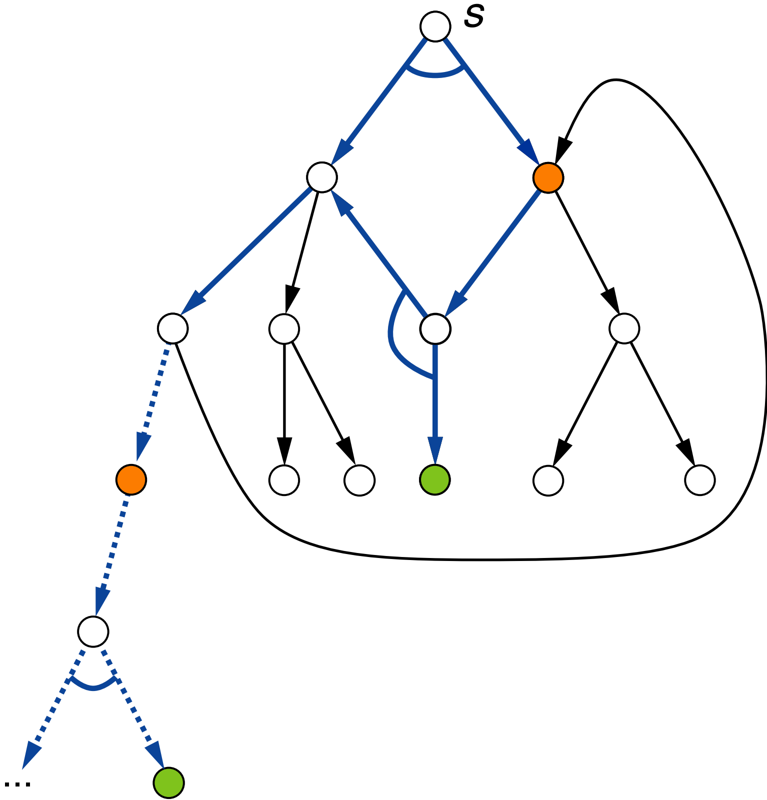
Problem-Reduction Representation

Solution Graphs with Cycles (1)

Cyclic problem-reduction graphs can entail infinite solution graphs, caused by "unfolding":



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Remarks:

- “Unfolding” means that when a node is reached a second time, the state is not reused but a copy is created instead. I.e., we are completely omitting an occurrence check.
- Note that the solution graph definition restricts to acyclic AND-OR graphs.

  Q. How could the solved-labeling procedure be adapted to deal with cyclic AND-OR graphs?
- Usually the search space underlying a problem is only partially explored, and hence the AND-OR graph H does not show the complete picture.

  Q. How could the solved-labeling procedure be adapted to deal with incomplete AND-OR graphs?
- Q. How could an “unsolvable-labeling procedure” be defined, which proves whether a problem is unsolvable?

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Problem-Reduction Representation

Hypergraphs

AND-OR graphs can be considered as a generalization of ordinary graphs, called *hypergraphs*:

- Ordinary graph: Directed links (edges, arcs) connect two nodes.
- Hypergraph: Directed hyperedges, also called “connectors”, connect two *sets* of nodes.
- AND-OR graph: The links of an AND node form a single hyperedge.

➜ The determination of a solution graph corresponds to the determination of a hyperpath between the root node s and a set of goal nodes that represent solved rest problems.

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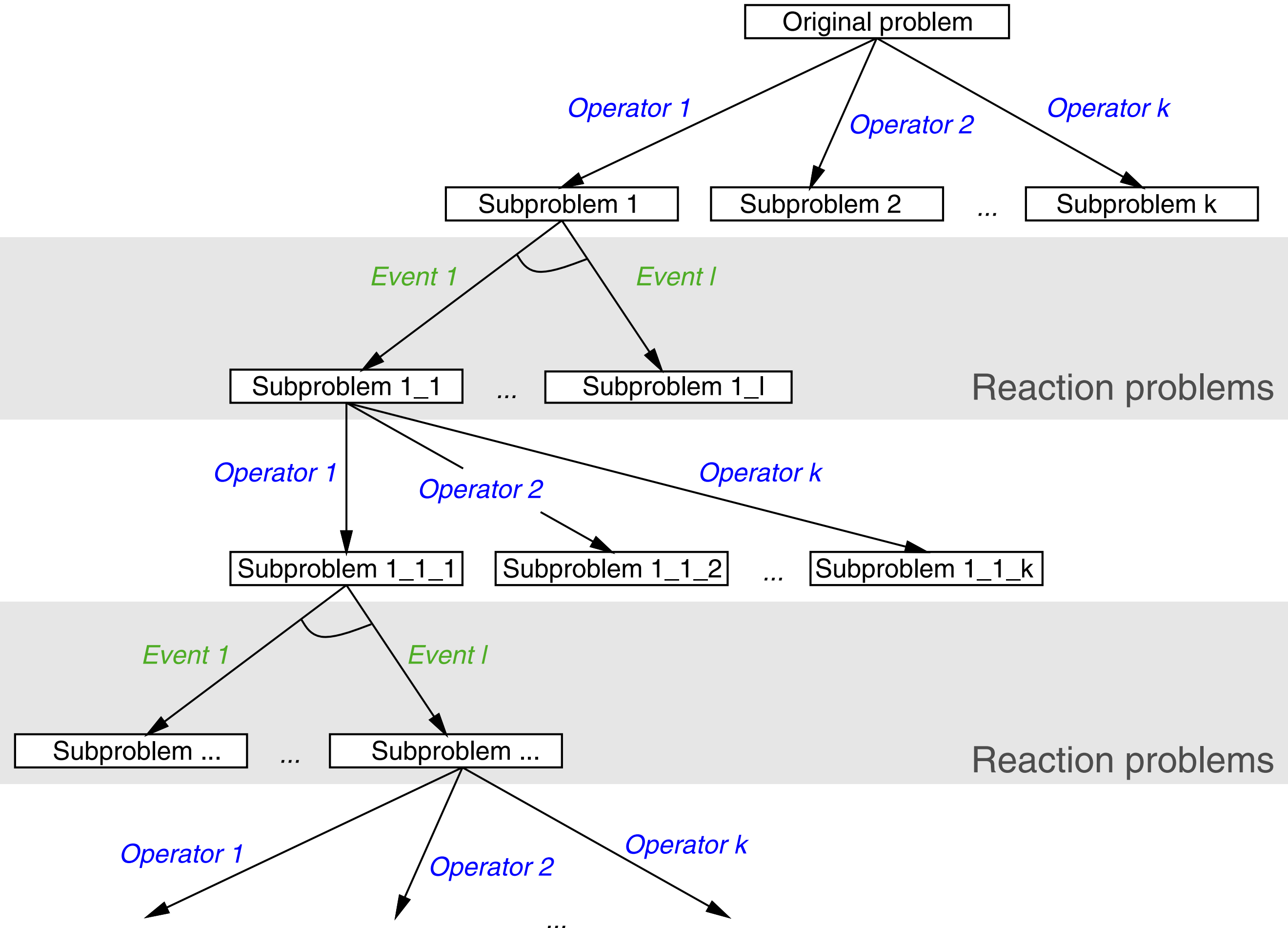
Choosing a Representation

Problem-Reduction Graphs with Alternating Node Types



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Remarks:

- By introducing OR nodes with a single operator resp. by introducing additional AND nodes with a single event, each AND-OR graph can be transformed into a problem-reduction graph with alternating node types. W.l.o.g. the start node of such graphs is an OR node.

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Choosing a Representation

Problem-Reduction Graphs with Alternating Node Types (continued)

Interpretation:

- k operators $\sim$ own actions, decisions, strategy
- l events $\sim$ reactions

- Action problems.
  Consequences that result from a subproblem under the impact of applicable operators.
- Reaction problems.
  Consequences that result from a subproblem under the impact of possible events.

Examples of problem-reduction graphs with strictly alternating node types are balance puzzles and 2-player games like chess.

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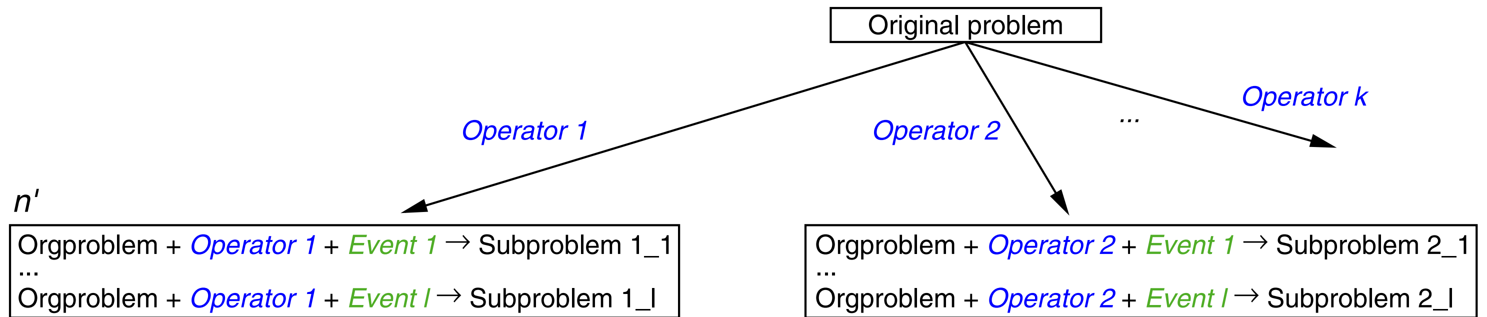
Choosing a Representation

Transforming Problem-Reduction into State-Space Graphs



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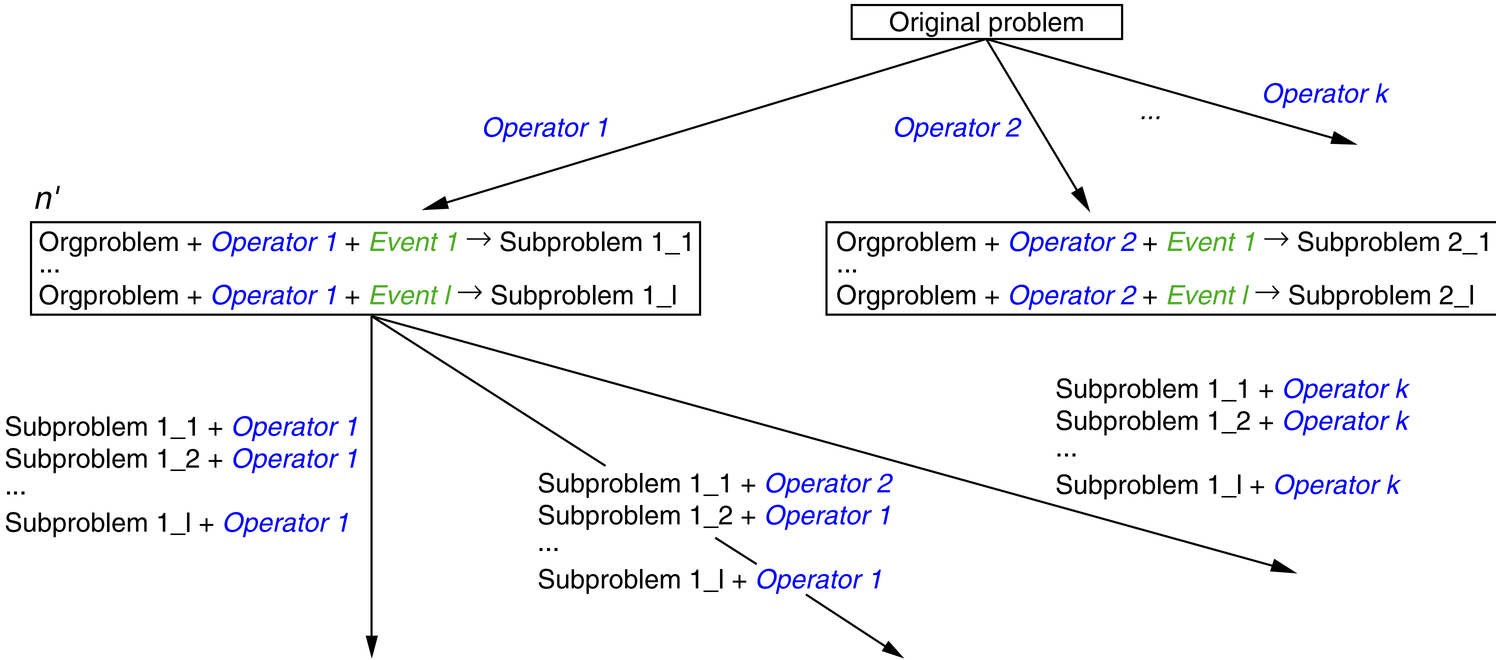
Choosing a Representation

Transforming Problem-Reduction into State-Space Graphs



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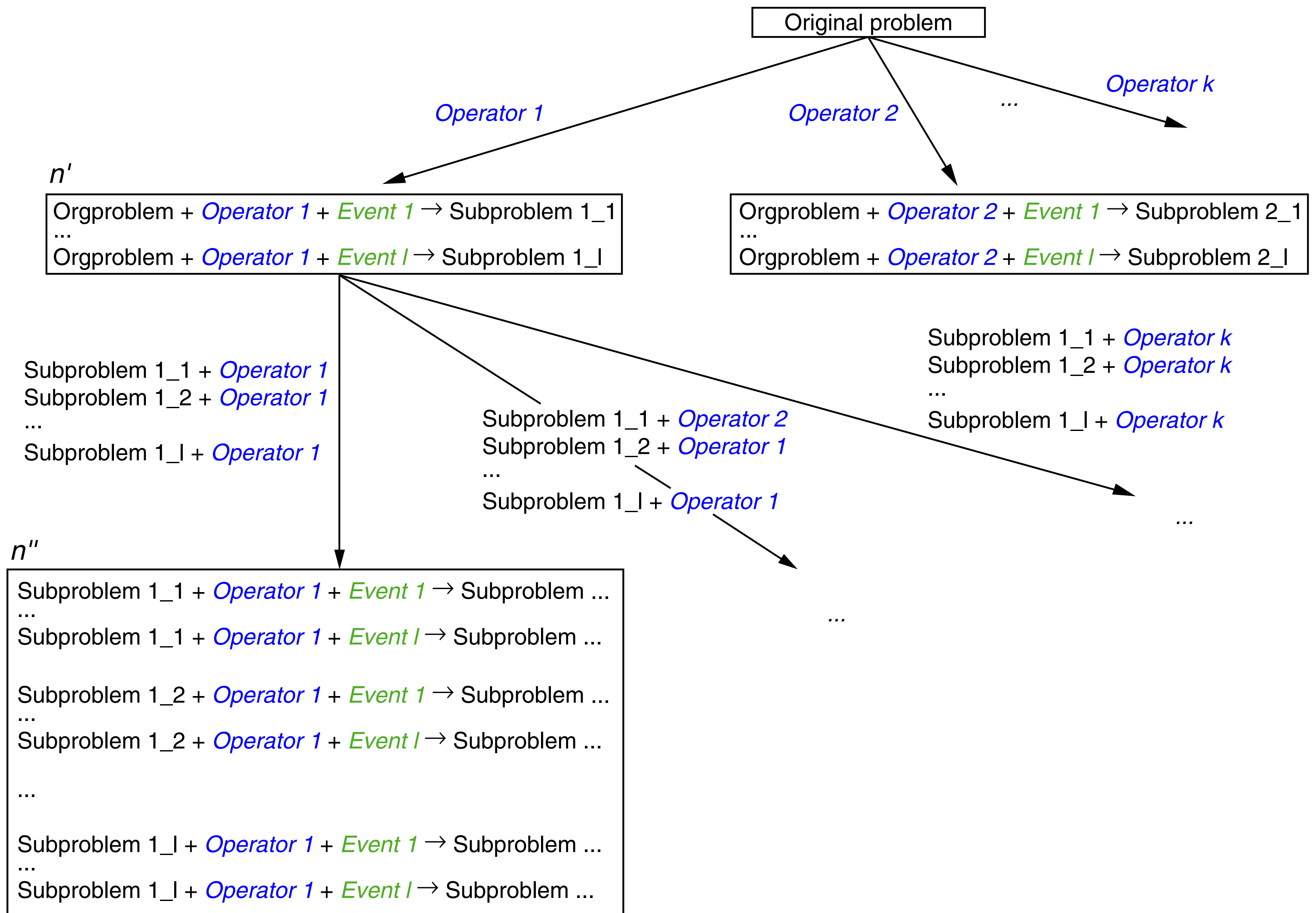
Choosing a Representation

Transforming Problem-Reduction into State-Space Graphs



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Choosing a Representation

Transforming Problem-Reduction into State-Space Graphs (continued)

Transformation rules:

- To the node s the applicable operators are applied.
- A successor node n' of s is comprised of all reaction problems that may result from applying the chosen operator.
- To a node n' all applicable operator combinations are applied, whereas each subproblem $p \in n'$ gets its own operator, i.e., a set of $|n'|$ operators is associated with the link (n', n'').
- The successor node n'' of n' is comprised of all reaction problems that may result from applying the chosen operator set.

  Observe that the node n' has as many successors as operator combinations are possible, given the subproblems $p \in n'$. This number is in $O(k^{|n'|})$

➜ Goal nodes are nodes that contain only trivial (solved) subproblems.

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Remarks:

- If a subproblem in a node is already solved, no further operator is applied to it.
- Instead of handling all subproblems in a node simultaneously, one subproblem could be selected at a time and an operator applied to it.

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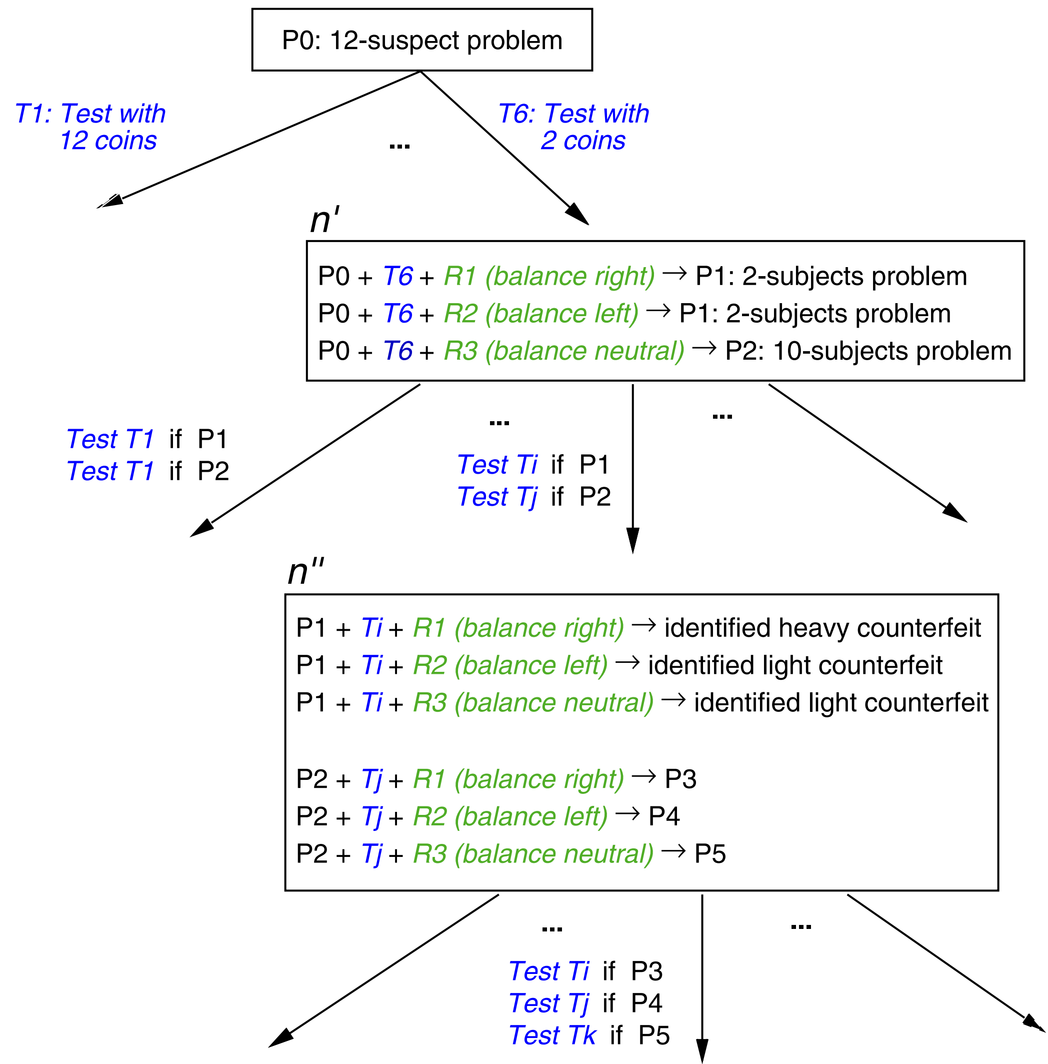
Choosing a Representation

Transformed Counterfeit Problem



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Choosing a Representation

State-Space versus Problem-Reduction Representation

A state-space representation is advisable if solutions are paths (or nodes).

➜ The solution graph is a path in the state-space graph.

Examples:

- constraint satisfaction problems
- sequence seeking problems

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Choosing a Representation

State-Space versus Problem-Reduction Representation (continued)

A problem-reduction representation is advisable if solutions are trees, graphs, or partially ordered node sets.

➜ The solution graph is a tree-like subgraph of the problem-reduction graph.

Examples:

- strategy seeking problems
- symbolic integration
- theorem proving

Characteristics:

- Problems may be decomposed into subproblems that can be solved independently of each other (= in parallel).
- The divide-and-conquer paradigm can be applied to find a global optimum.

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Remarks:

- Choosing a search space representation requires the analysis whether and how the solutions of subproblem interact with each other when being composed.
- Q. Is a problem-reduction representation possible for the 8-puzzle problem?
- Q. Is a problem-reduction representation an advisable choice for the 8-puzzle problem?
- There are sequence seeking problems where the solutions of subproblems interact with each other (e.g., the 8-puzzle problem), but where a problem-reduction representation is superior to a state-space representation. Example: The tower of Hanoi problem.

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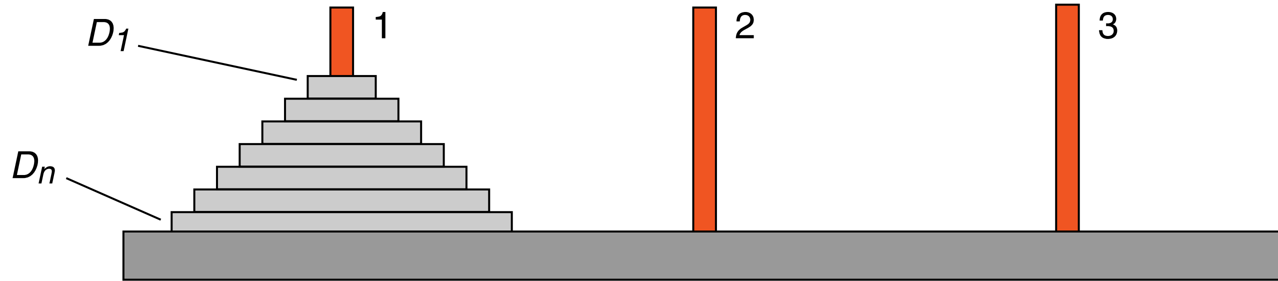
Choosing a Representation

Tower of Hanoi Problem [Wikipedia]



A stack of n disks $D_1, \ldots, D_n$ (different sizes, sorted, smallest on top) is to be moved from peg 1 to peg 3. Only one of the topmost disks can be moved at a time, no disk may be placed on top of a smaller one.

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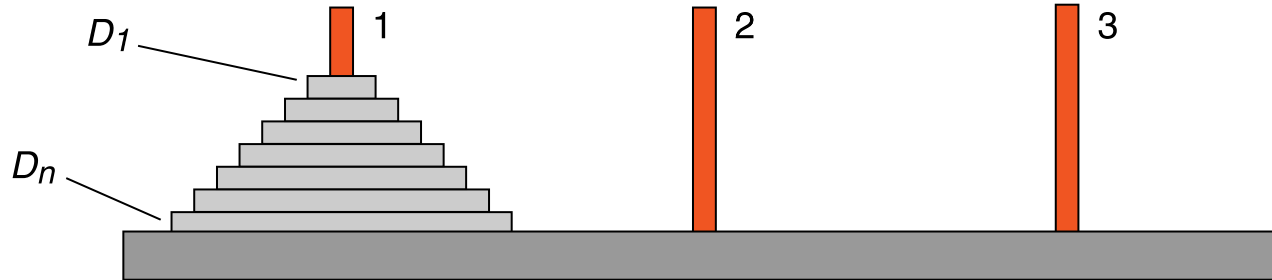
Choosing a Representation

Tower of Hanoi Problem [Wikipedia]



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Observations:

- The problem can be solved analogously to solving the 8-puzzle problem. A sequence of legal moves of single disks has to be determined.
- The recursive definition of moving a stack of disks $D_1, \ldots, D_k$ from one peg to another peg can be exploited: problem decomposition.

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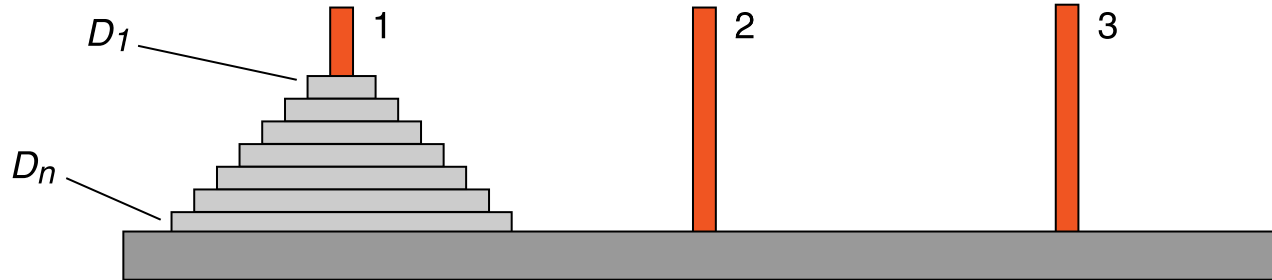
Choosing a Representation

Tower of Hanoi Problem [Wikipedia]



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Observations:

- The problem can be solved analogously to solving the 8-puzzle problem. A sequence of legal moves of single disks has to be determined.
- The recursive definition of moving a stack of disks $D_1, \ldots, D_k$ from one peg to another peg can be exploited: problem decomposition.

→ Moving a stack of disks $D_1, \ldots, D_k$ from peg a to peg b can be done in three “steps”:

1. Move the stack of disks $D_1, \ldots, D_{k-1}$ from peg a to peg c.
2. Move disk D_k peg a to peg b.
3. Move the stack of disks $D_1, \ldots, D_{k-1}$ steps peg c to peg b.

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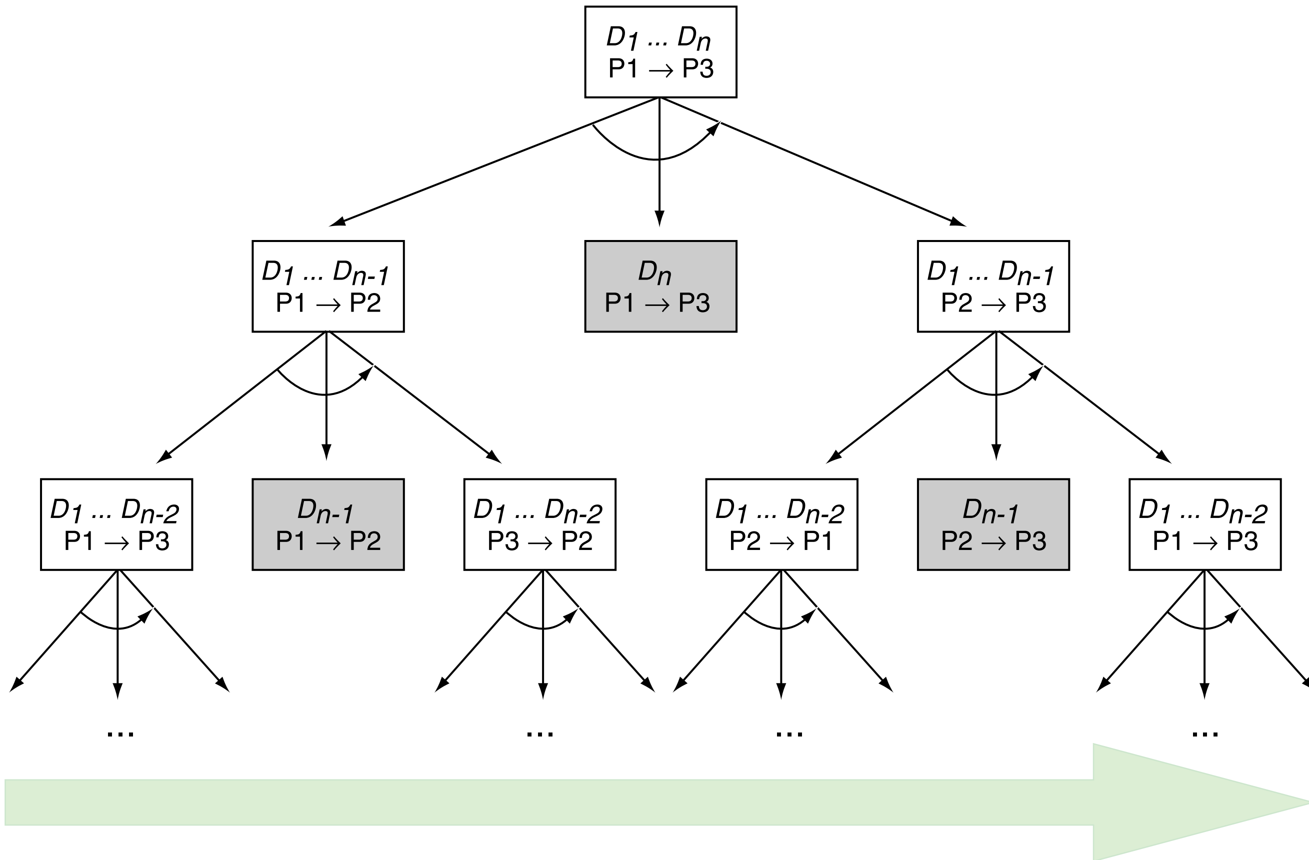
Choosing a Representation

Tower of Hanoi Problem (continued)



➜ Problem decomposition can be a powerful tool, even if order constraints have to be taken into account.

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Choosing a Representation

Tower of Hanoi Problem (continued)

A stack of n disks $D_1, \ldots, D_n$ (different sizes, sorted, smallest on top) is to be moved from peg 1 to peg 3. Only one of the topmost disks can be moved at a time, no disk may be placed on top of a smaller one.

OR Graph Representation:

- Nodes are states (i.e., the stacked disks for each peg). The remaining problem is known.
- Edges correspond to legal moves of some single disk.

→ Cyclic paths can be pruned.

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Choosing a Representation

Tower of Hanoi Problem (continued)

A stack of n disks $D_1, \ldots, D_n$ (different sizes, sorted, smallest on top) is to be moved from peg 1 to peg 3. Only one of the topmost disks can be moved at a time, no disk may be placed on top of a smaller one.

AND-OR Graph Representation:

- ❑ Nodes are states (i.e., the stacked disks for each peg). The remaining problem has to be stored explicitly.
- ❑ All nodes are AND nodes.
- ❑ Families of AND edges correspond to the following problem decomposition:
  - – hardest single move problem,
    (Moving the biggest misplaced disk from its current position to its target position.)
  - – establishing a state that allows that move,
    (Moving all disks on top of the biggest misplaced disk out of the way.)
  - – establishing the target state out of the state resulting from that move.
    (Moving the remaining misplaced disks to their target psotions.)

➜ A solution graph will be constructed directly.

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Choosing a Representation

Tower of Hanoi Problem (continued)

A stack of n disks $D_1, \ldots, D_n$ (different sizes, sorted, smallest on top) is to be moved from peg 1 to peg 3. Only one of the topmost disks can be moved at a time, no disk may be placed on top of a smaller one.

AND-OR Graph Representation: (continued)

- ➜ Start with the most difficult problem: D_n from peg 1 to peg 3.
  - ➜ D_n must be clear and peg 3 must be empty.
    - ➜ Introduce new subproblems: $D_1, \ldots, D_{n-1}$ from peg 1 to peg 2.
- ➜ Solve the remaining problems.

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Choosing a Representation

Means-End Analysis

Characteristic for the tower of Hanoi problem is that the set of subproblems $p_1, \ldots, p_n$ is serializable. Formally, this means that there is a sequence $i_1, \ldots, i_n$ such that for problems $p_{i_1}, \ldots, p_{i_n}$:

1. $p_{i_1}, \ldots, p_{i_k}$ can be solved after $p_{i_{k+1}}, \ldots, p_{i_n}$ have been solved.
2. Solving $p_{i_1}, \ldots, p_{i_k}$ will not compromise the solutions of $p_{i_{k+1}}, \ldots, p_{i_n}$.

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Choosing a Representation

Means-End Analysis

Characteristic for the tower of Hanoi problem is that the set of subproblems $p_1, \ldots, p_n$ is serializable. Formally, this means that there is a sequence $i_1, \ldots, i_n$ such that for problems $p_{i_1}, \ldots, p_{i_n}$:

1. $p_{i_1}, \ldots, p_{i_k}$ can be solved after $p_{i_{k+1}}, \ldots, p_{i_n}$ have been solved.
2. Solving $p_{i_1}, \ldots, p_{i_k}$ will not compromise the solutions of $p_{i_{k+1}}, \ldots, p_{i_n}$.

If this underlying problem structure is identified, the powerful operator selection strategy “means end analysis” can be applied:

> *“The basic difference between this method* [means end analysis] *and the state-space approach is its purposeful behavior: operators are invoked by virtue of their potential in fulfilling a desirable subgoal, and new subgoals are created in order to enable the activation of a desirable operator.”* [Pearl 1984, p.29]

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Remarks:

- Planning a trip (e.g. to New York) is another example for a problem that can be tackled by a means end analysis:
  1. New York $\rightarrow$ plane from Frankfurt
  2. plane from Frankfurt $\rightarrow$ train to Frankfurt
  3. train to Frankfurt $\rightarrow$ bus to train station
  4. ...
- The power of problem reduction representation under a regime of a means end analysis: AND-linked subproblems can be solved independently, though for processing the global plan a strict order (from left to right) must be obeyed.
- Q. What is the complexity class of the tower of Hanoi problem?

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Chapter S:IV

IV. Search Space Representation

- ❑ Problem Solving
- ❑ Systematic Search
- ❑ Search Space Encoding
- ❑ State-Space Representation

- ❑ Problem-Reduction Representation
- ❑ Choosing a Representation
- ❑ Relation to Dynamic Programming

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Relation to Dynamic Programming

Optimum Solution Cost and Bellman's Principle of Optimality

Dynamic programming has been developed to solve

- discrete optimization problems and
- planning problems for which an optimum sequence of decisions is sought.

Bellman's principle of optimality:

> *"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."*

[Bellman 1954]

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> [Bellman 1954]

A stronger version of Bellman’s optimality principle:

> *Each partial solution of an optimum solution is in turn an optimum solution for the respective subproblem.*

➜ Optimum (estimated) solution cost can be defined via systems of equations.

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Relation to Dynamic Programming

Optimum Solution Cost and Bellman's Principle of Optimality (continued)

Definition 6 (Bellman Equations for C^* [Overview])

Let G be an acyclic AND-OR graph rooted at s. Let $C_H(n)$ be a recursive cost function for G based on E (local properties).

$$C^*(n) = \begin{cases} c(n) & n \text{ is goal node and leaf in } G \\ \infty & n \text{ is leaf in } G \text{ but no goal node} \\ \min_i\{F[E(n), C^*(n_i)]\} & n \text{ is inner OR node in } G, \\ & n_i \text{ direct successor of } n \text{ in } G \\ F[E(n), C^*(n_1), \ldots, C^*(n_k)] & n \text{ is inner AND node in } G, \\ & n_i \text{ direct successor of } n \text{ in } G \end{cases}$$

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If F is properly chosen ("If F obeys Bellman's principle of optimality"), the equations stated in Definition 6 form necessary and complete conditions for the optimality of the solution cost C^*.

"$\Rightarrow$" The values that fulfill the Bellman equations are the optimum solution cost (for the respective node).

"$\Leftarrow$" The optimum solution cost for a node fulfill the Bellman equations.

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Remarks:

- The name “dynamic programming” was introduced by Bellman. As with “linear programming”, the terms “program” and “programming” should be read as “plan” and “plan generation” respectively.

- The term “policy” characterizes the process of decision making and corresponds to the principle of selecting a most promising solution base.

- The objective function in discrete optimization problems corresponds to the evaluation functions f (state-space search) or f_1 (problem-reduction search).

- Observe the rationale behind the Bellman equations (Definition 6) and optimality:

  1. Given a problem (modeled as AND-OR graph) we “may be lucky” that its solutions obey Bellman’s principle of optimality (consider that we have no free choice of F).
     Similarly: The cost measure F imposed by the problem obeys Bellman’s principle of optimality (examples for F include “$+$” and “$\max$”). In this case we should formulate the optimization (= cost) function according to Definition 6.

  2. Then, operationalizing Definition 6 will yield the optimum solution cost for our problem.

  Conversely, the fact that the solutions of our problem won’t obey Bellman’s principle of optimality means that we cannot formulate its cost function recursively *and* equip it with an amenable cost measure (examples for F include “$+$” and “$\max$”). We still may be able to compute a cost function as solution of Bellman’s equations according to Definition 6, but this operationalization will probably not yield the optimum solution cost for our problem.

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Using the face-value principle for partially explored search spaces, Bellman's principle of optimality induces a system of equations defining estimated optimum solution cost $\widehat{C}(n)$.

Definition 7 (Bellman Equations for $\widehat{C}$ [Overview, monotone F] **)**

Let G be an explored subgraph of an acyclic AND-OR graph $\mathcal{G}$ rooted at s. Let h be an underestimating heuristic function, let $C_H(n)$ be a recursive cost function for $\mathcal{G}$ based on E and F, and let F obey Bellman's principle of optimality.

$$\widehat{C}(n) = \begin{cases} c(n) & n \text{ is goal node and leaf in } G \\ h(n) & n \text{ is leaf in } G \text{ but no goal node} \\ \min_i\{F[E(n), \widehat{C}(n_i)]\} & n \text{ is inner OR node in } G, \\ & n_i \text{ direct successors of } n \text{ in } G \\ F[E(n), \widehat{C}(n_1), \ldots, \widehat{C}(n_k)] & n \text{ is inner AND node in } G, \\ & n_i \text{ direct successors of } n \text{ in } G \end{cases}$$

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Relation to Dynamic Programming

Optimum Solution Cost and Bellman's Principle of Optimality (continued)

Corollary 8 (Bellman Equations for $\widehat{C}$)

A recursive cost function $C_H(n)$ that is based on local properties E and a monotone cost measure F obeys Bellman's principle of optimality.

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Relation to Dynamic Programming

Partially Explored State Spaces

Definition 9 (Bellman Equations for $\widehat{C}$ [Overview, monotone F]**)**

Let G be an explored subgraph of a state-space graph $\mathcal{G}$ rooted at s. Let h be an underestimating heuristic function, let $C_P(n)$ be a recursive cost function for $\mathcal{G}$ based on E and F, and let F obey Bellman's principle of optimality.

$$\widehat{C}(n) = \begin{cases} c(n) & n \text{ is goal node and leaf in } G \\ h(n) & n \text{ is leaf in } G \text{ but no goal node} \\ \min_i\{F[E(n), \widehat{C}(n_i)]\} & n \text{ is inner OR node in } G, \\ & n_i \text{ direct successors of } n \text{ in } G \end{cases}$$

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Relation to Dynamic Programming

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Special case A*:

- use top down propagation of path cost values $g(n)$
- consider only paths to nodes in OPEN

$$\widehat{C}(n) = \begin{cases} h(n) & n \text{ is on OPEN} \\ \min_i\{c(n, n_i) + \widehat{C}(n_i)]\} & n \text{ is on CLOSED, } n_i \text{ direct successors of } n \end{cases}$$

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Relation to Dynamic Programming

Partially Explored State Spaces (continued)

Algorithm A* and Bellman's principle of optimality:

1. Top-down Propagation.

   Superior (optimum) paths to non-goal states are extended to superior (optimum) paths to successor states until a goal is reached.

➜ Right-Monotonicity $\sim$ Order Preservation.

For all nodes n' and paths $P_{s-n'}$, $P'_{s-n'}$ from s to n', and for all nodes n and paths $P_{n'-n}$ from n' to n holds:

$$\widehat{C}_{P_{s-n'}}(s) \leq \widehat{C}_{P'_{s-n'}}(s) \quad \Rightarrow \quad \widehat{C}_{P_{s-n}}(s) \leq \widehat{C}_{P'_{s-n}}(s)$$

where P_{s-n} uses subpath $P_{s-n'}$ and P'_{s-n} uses $P'_{s-n'}$.

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Relation to Dynamic Programming

Partially Explored State Spaces (continued)

Algorithm A* and Bellman's principle of optimality:

2. Bottom-up Propagation.

   Starting from optimum solutions for rest problems, optimum solutions for complexer problems are synthesized until the original problem is solved.

➜ Left-Monotonicity.

For all nodes n' and paths $P_{n'-\gamma}$, $P'_{n'-\gamma}$ from n' to γ, and for all paths $P_{s-n'}$ from s to n' holds:

$$\widehat{C}_{P_{n'-\gamma}}(n') \leq \widehat{C}_{P'_{n'-\gamma}}(n') \quad \Rightarrow \quad \widehat{C}_{P_{s-\gamma}}(s) \leq \widehat{C}_{P'_{s-\gamma}}(s)$$

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